Variational Analysis and Optimization (VAO 2024)

In honor of Nicolas HADJISAVVAS

The role of asymmetry in applied analysis

Aris DANIILIDIS

TU Wien

Milano, May, 30, 2024

Based on the work:

— A. Daniilidis, J.M. Sepulcre, F. Venegas M.

Asymmetric free spaces and canonical asymmetrizations

Studia Mathematica (2021)

Part of the PhD Thesis of Francisco Venegas:

Functional Analysis in asymmetric structures

PhD Thesis, University of Chile (2023)

Breaking the symmetry is paramount in several instances:

– modeling complex phenomena
(Finsler manifolds, oriented graphs...)

- remedy lack of smoothness
(convexity, subdifferential theory;
PDE / theory of viscosity solutions)

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M. Gromov on requiring symmetry in the definition of metrics:

"...This unpleasantly limits many applications: the effort of climbing up to the top of a mountain, in real life as well as in mathematics, is not at all the same as descending back to the starting point."

Intro of the book
"Metric Structures for Riemannian and Non-Riemannian Spaces"

Functional analysis:

Ş. Cobzaş, *Functional Analysis in Asymmetric Normed Spaces*, Birkhäuser, 2012

Metric analysis

Cabello-Sanchez, Garrido, Jaramillo ...

Akian, Gaubert, Vigeral

Gromov compactification, Martin boundary, Oriented horofunctions

Finsler geometry:

D. Bao, S-S. Chern, Z. Shen, An Introduction to Riemann-Finsler Geometry, vol. 200, Springer Science & Business Media, 2012.

M.A. Javaloyes, L. Lichtenfelz, P. Piccione, Almost isometries of non-reversible metrics with applications to stationary spacetimes, J. Geom. Phys. 89 (2015) 38–49.

...

Viscosity solutions:

(Partial Differential Equations)

Instead of solving $F(x, u(x), Du(x), D^2u(x)) = 0$

we solve $F(x, \phi(x), D\phi(x), D^2\phi(x)) \leq 0$, for $\phi \geq u$

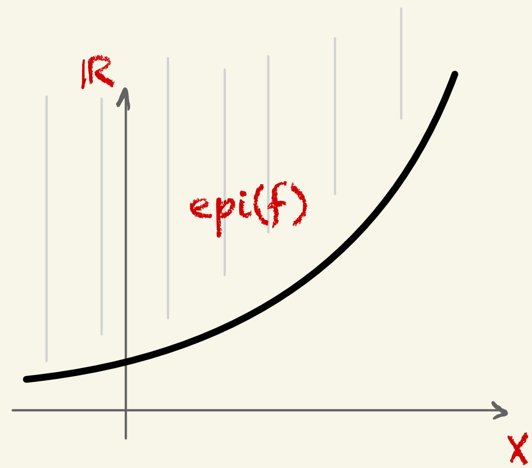
and $F(x, \psi(x), D\psi(x), D^2\psi(x)) \geq 0$, for $\psi \leq u$

(around x), $\phi, \psi \in C^2$

Convex analysis:

convex function:

$\mathrm{epi}(f) \subset X \times \mathbb{R}$ convex



$\partial f(x) = \{ p \in X^* : f - p$ attains a minimum at $x \}$

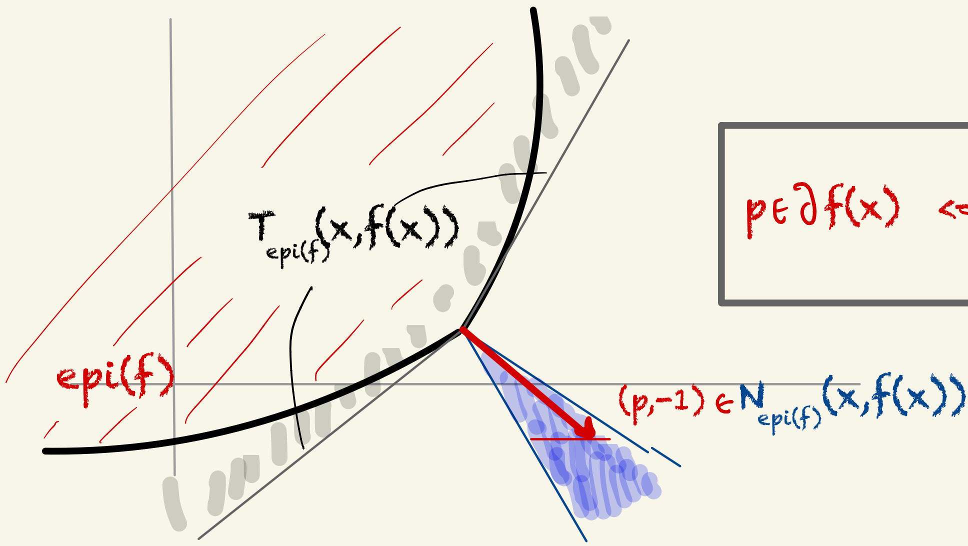
Nonsmooth analysis (Unilateral analysis)

Replace: graphs by epigraphs

tangent spaces by tangent cones

normal spaces by normal cones

derivatives (gradients) by subdifferentials (subgradients)



$$p \in \partial f(x) \iff (p,-1) \in N_{epi(f)}(x,f(x))$$

Typical non-symmetric objects

$\{a\}^+ := \max\{a,0\}$ (asymmetric hemi-norm in $\mathbb{R}$)

$\|h\|_{\infty,\mathcal{U}} = \sup_{x\in\mathcal{U}} \{h(x)\}^+$ (asymmetric uniform norm on $\mathcal{U}$)

Metric slope: $$s_f(x) = \liminf_{y\to x} \frac{\{f(x)-f(y)\}^+}{d(y,x)}$$ (De Giorgi, 1980)

- if f is convex, then $s_f(x) = d(0, \underbrace{\partial f(x)}_{\text{subdifferential}})$

Meta-theorem (sensitivity type result)

If the slopes of two convex functions are close, then the functions should also be close (in some sense).

Let f, g be convex continuous functions in $\mathbb{R}$.

Set $C_f = \operatorname{argmin}(f)$ and $U_r = \{ x \in \mathbb{R} : d(x, C_f) \lesssim r \}$

Then

$$\|g - f\|_{\infty, U_r} \lesssim \|s_g - s_f\|_{\infty, U_r} + \|g - f\|_{\infty, U_r}$$

Daniilidis-Drusviatskiy

Proc. Amer. Math. Soc., (2023)

Set-valued analysis (coderivatives)

$T: X \rightrightarrows Y$ (convex) positively homogeneous $\begin{cases} (T(x+y) \subset T(x)+T(y)) \\ T(\lambda x) = \lambda T(x), \quad \lambda > 0 \end{cases}$

$$\|T\|^+ = \sup_{\|x\|=1} \sup_{y \in T(x)} \|y\|.$$

norm

Given $F: X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$, we define the co-derivative:

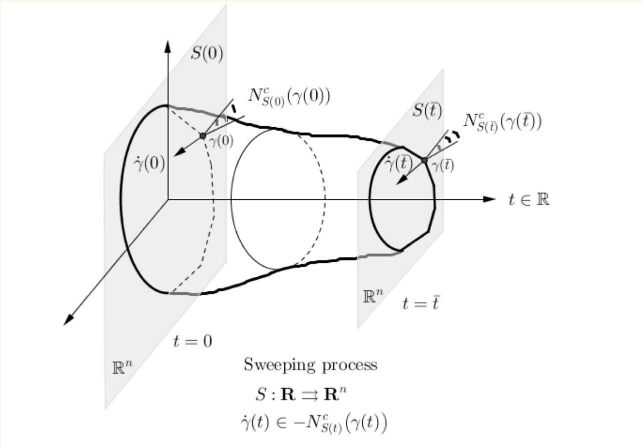
$T = D^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ through the relation:

$$(y^*, x^*) \in \operatorname{gph}(T) \iff (x^*, -y^*) \in N_{\operatorname{gph}(F)}(x, y)$$

If $Y = \mathbb{R}$ and $F(x) = [f(x), +\infty)$ where f is convex (resp. smooth)

then $D^* F(x, f(x)) = \mathbb{R}_+ \{(1, p) : p \in \partial f(x)\}$ (resp. $\mathbb{R}_+ (1, Df(x))$)

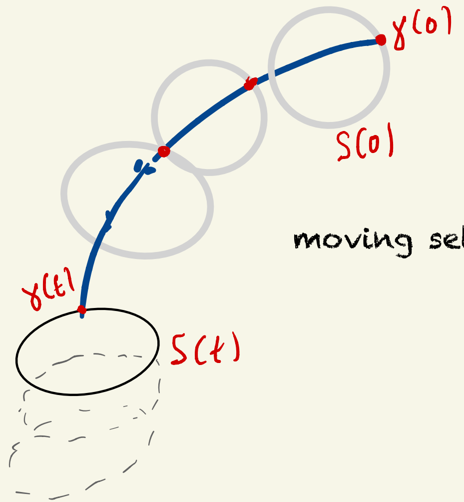
Case $X = \mathbb{R}$: (sweeping process dynamics)



$S: \mathbb{R} \rightrightarrows \mathbb{R}^n$

$\dot\gamma(t) \in -N_{S(t)}(\gamma(t))$

$\gamma(0) \in S(0)$



$$\|D^*S(t,x)\|^+ := \sup_{\|u\|\leq 1} \Big\{ |a| : a \in D^*S(t,x)(u) \Big\}.$$

modulus of coderivative
(tame sweeping process)

$$\|D^*S(t,x)|^{+,\uparrow} = \sup_{\|u\|\leq 1} \Big\{ a^+ : \ a \in D^*S(t,x)(u) \Big\},$$

asymmetric modulus of coderivative
(adapted for the asymptotic analysis of a general process)

Daniilidis, Tapia-Garcia
Math. Oper. Res. (2023)

Quasi-metric space

(X,d) is called quasi-metric space if

(i). $d(x,z) \leq d(x,y) + d(y,z)$ for all x,y,z in X

(ii). $x=y$ if and only if $d(x,y)=0$ — d quasi-distance

or

(ii)'. $x=y$ if and only if $\begin{cases} d(x,y)=0 \\ d(y,x)=0 \end{cases}$ — d quasi-hemi-distance

(iii). $d(x,x)=0$ and $d \geq 0$

The definition allows $d(x,y) \neq d(y,x)$ and one of the two can be 0 (for quasi-hemi-distances).

Index of asymmetry: $\sup_{x \neq y} \frac{d(x,y)}{d(y,x)} - 1 \in [0,+\infty]$

Notice that

$$B^+(x,r) := \{y \in X : d(x,y) < r\} \neq B^-(x,r) := \{y \in X : d(y,x) < r\}$$

forward ball (left), backward ball (right)

$\mathcal{T}_+$ forward topology (generated by $\{B^+(x,r) : x \in X,\ r > 0\}$)

$\mathcal{T}_-$ backward topology (generated by $\{B^-(x,r) : x \in X,\ r > 0\}$)

The above topologies are always first countable and T_0 (but possibly not T_1)

If the index of asymmetry is not $+\infty$ then both topologies coincide with the metric topology of the symmetrized distance.

Convergence (forward topology)

We say that $x_n \xrightarrow{\mathcal{T}_+} x$ if $d(x,x_n) \xrightarrow[n\to\infty]{} 0$. (The limit might not be unique)

Example (non-uniqueness of forward limit)

Let $\{x_n\}_{n\geq 1}$ be a sequence of distinct elements and let $\bar{x}$, $\bar{y}$ be distinct to each other and to any element of the sequence x_n

Define $d(\bar{x}, x_n) = d(\bar{y}, x_n) = 1/n$ and $d(x, y) = 1$ for any other case where $x \neq y$.

Then $x_n \xrightarrow{\mathcal{T}_+} \bar{x}$ and $x_n \xrightarrow{\mathcal{T}_+} \bar{y}$.

Notice that the symmetrized distance

$$D^s(x,y) = \max\{d(x,y), d(y,x)\}$$

is discrete

Example

$(\mathbb{R}, d_u)$ $\quad d_u(x,y) = \{y-x\}^+ = \begin{cases} y-x, & \text{if } y \geq x \\ 0, & \text{if } y < x \end{cases}$

The forward topology is generated by the (forward) balls

$B^+(x,r) = (-\infty, x+r)$, for every $x \in X$, $r > 0$.

Proposition

The quasi-metric d_u **characterizes upper semi-continuity***: a function from a topological space $f : (X, \tau) \to \mathbb{R}$ is continuous for the forward topology of $(\mathbb{R}, d_u)$ if and only if f is upper semi-continuous for the usual topology on $\mathbb{R}$*

Asymmetric normed space

$(X, \|\cdot|)$ is called asymmetric normed space if:

(i). $\|x+y| \le \|x| + \|y|$, for all x, y in X

(ii). $\|\lambda x| = \lambda \|x|$, for all $x \in X$ and $\lambda > 0$.

(iii). $x=0$ if and only if $\|x|=0$ — $\|\cdot|$ asymmetric norm

or

(iii)'. $x=0$ if and only if $\begin{cases} \|x| = 0 \\ \|-x| = 0 \end{cases}$ — $\|\cdot|$ asymmetric hemi-norm

---

It is not a topological vector space (in general).

Finite asymmetry, if $\exists M > 0$ such that $\|-x| \le M \|x|$, for all $x \in X$.

↳ $\|\cdot|$ defines the same topology as the symmetrized norm $\|x\|_s := \max\{\|x|, \|-x|\}$

Given an asymmetric (hemi) norm we can define an asymmetric (hemi) distance as follows:

$$d(x,y) := \|y-x|$$

– Hahn-Banach theorem still holds (!)

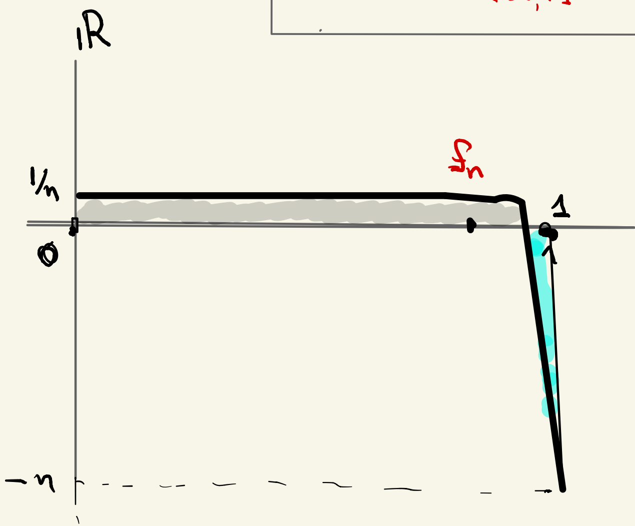
- The dual space (= space of $\|.|$-bounded linear functionals) is no longer a linear space (it is a normed cone instead!)

– Completeness can be defined through the symmetrized norm (Bi-completion)

Example:

$$X = \{ f \in C([0,1]) : \int_0^1 f(t)\,dt = 0 \}$$

$$\|f\|_{\infty,+} := \max_{t\in[0,1]} \{f(t)\}^+ = \|f^+\|_\infty$$



It is possible to have:

$\{f_n\}_n \subset X$

$\|f_n\|_\infty = \frac{1}{n} \to 0$ and

$\|-f_n\|_\infty = n \to \infty$

$\delta_1 : X \to \mathbb{R}$

$\delta_1(f) = f(1)$

$$\begin{cases} \delta_1(f_n) = 0 \\ \delta_1(-f_n) = -n \to -\infty \end{cases}$$ (not l.s.c!)

$\delta_1 \in X^*$

but $-\delta_1 \notin X^*$

Need to consider the notion of normed cone

Definition (Abstract cone)

A **cone** on $\mathbb{R}_+$ is a triple $(C, +, \cdot)$ such that $(C, +)$ is an Abelian monoid, and $\cdot$ is a mapping from $\mathbb{R}_+ \times C$ to C such that for all $x, y \in C$ and $r, s \in \mathbb{R}_+$:

1. $r \cdot (s \cdot x) = (rs) \cdot x$,
2. $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$ and $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$,
3. $1 \cdot x = x$ and $0 \cdot x = 0$.

A (conic) norm $\|.\|$ is defined in a similar way as the asymmetric norm in linear spaces.

Proposition

Let $\|\cdot|$ be a conic-norm on a cone $(C,+,\cdot)$. Then the function d_e defined by

$$d_e(x,y)=\begin{cases}\inf\limits_{\substack{z\in C\\ y=x+z}}\|z|, & \text{if } \mathbf{y}\in\mathbf{x}+\mathbf{C}\\ +\infty, & \text{if } \mathbf{y}\notin\mathbf{x}+\mathbf{C}\end{cases}$$

is an **extended** *quasi-metric on C.*

If C is a linear space, then $d_E(x,y) = \|y - x|$.

Given an asymmetric norm $\|.|$ (resp. quasi-distance d)

we can define (canonical) symmetrizations as follows:

$$\|x\|_{s_0} := \max\{ \|x|, \|-x| \} \quad \text{or} \quad \|x\|_{s_1} := \|x| + \|-x\|$$

(respectively,

$$D_{s_0}(x,y) := \max\{d(x,y), d(y,x)\} \quad \text{or} \quad D_{s_1}(x,y) := d(x,y) + d(y,x) \;)$$

Question: what about (canonical) asymmetrizations ?

ASYMMETRIZATION

There is a natural way to asymmetrize the norm of the classical Banach spaces (ie. spaces of sequences or functions)

- For sequences $x = \{x_n\}_{n\geq 1}$

  replace $|x_n|$ by $\{x_n\}^+ := \max\{x_n, 0\}$ for all $n \in \mathbb{N}$

Examples: $\ell^1(\mathbb{N})$; $\|x\|_{1,+} = \sum_{n=1}^{\infty} \{x_n\}^+$; $c_0(\mathbb{N})$; $\|x\|_{0,+} = \max_{n\geq 1} \{x_n\}^+$

- For real-valued functions f

  replace $|f(x)|$ by $\{f(x)\}^+ := \max\{f(x), 0\}$ for all x

Examples: $L^1(\mu)$; $\|f\|_{1,+} = \int_{\Omega} \{f(\omega)\}^+ d\mu(\omega)$; $C([0,1])$; $\|f\|_{\infty,+} = \sup_{t\in[0,1]} \{f(t)\}^+$

Question: How to asymmetrize a general normed space ?

We know the answer for classical Banach spaces

Every metric space embeds isometrically into a Banach space:

Arens-Eells space, free Lipschitz space (Kalton-Godefroy)

(X,d) metric space, $x_0 \in X$

$\mathrm{Lip}_0(X) := \{\phi : X \to \mathbb{R},\ \text{Lipschitz},\ f(x_0)=0\}$,

$$\|\phi\|_L = \sup_{x \neq y} \frac{\phi(y)-\phi(x)}{d(y,x)}$$

$(\mathrm{Lip}_0(X), \|\cdot\|_L)$ Banach space.

$$X \underset{=1}{\hookrightarrow} (\mathrm{Lip}_0(X), \|\cdot\|_L)^*$$

$x \longmapsto \delta_x$ (Dirac)

$$\mathcal{F}(X) = \overline{\mathrm{span}\{\delta_x : x \in X\}}^{\|\cdot\|_*} \subseteq (\mathrm{Lip}_0(X))^*$$

$$\delta_x(\phi) \equiv \phi(x),\ \forall \phi \in \mathrm{Lip}_0(X)$$

$$d(x,y) = \|\delta_x - \delta_y\|_* := \sup_{\|\phi\|_L \leq 1} [\phi(x) - \phi(y)]$$

Take as metric space our Banach space $(X, \|\cdot\|)$ and base point $x_0 = 0$.

Set $L := \mathrm{Lip}(X)$ and denote by $\langle \cdot, \cdot \rangle$ the duality map of the pair $\langle L, L^* \rangle$

Then:

$$\|Q\|_{\mathcal{F}} := \sup_{\substack{\phi \in L \\ \|\phi\|_L \leq 1}} \langle \phi, Q \rangle, \quad \text{for every } Q \in \mathcal{F}(X).$$

Let P be a generating (closed convex) cone of L (ie. $X = P - P$)

such that: $\forall \phi \in L,\ \exists \phi_1, \phi_2 \in P :$

$$\begin{cases} \phi = \phi_1 - \phi_2, \\ \max\{\|\phi_1\|_L, \|\phi_2\|_L\} \leq \|\phi\|_L \leq \|\phi_1\|_L + \|\phi_2\|_L. \end{cases}$$

We set:

$$\|Q|_{\mathcal{F}_P} := \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} \langle \phi, Q\rangle \qquad \text{for every } Q\in\mathcal{F}(X)$$

P-asymmetrization of $\mathcal{F}=\mathcal{F}(X)$

We call the asymmetrization canonical is P is of the form:

$$P := \{\phi\in L : T\phi \ge 0\}$$

where T is a lineal asymmetry that identifies L ($=\mathrm{Lip}(X)$) with some Banach lattice in a natural way. 0

Example:

$X=\mathbb{R}$; $L=\mathrm{Lip}_0(\mathbb{R}) =_1 L^\infty(\mathbb{R})$; P → $\{\phi\in L : \phi\ge 0\} = L_+$; P → $\{\phi\in L : \phi'\ge 0\}$

Asymmetrization of Lipschitz free spaces

↓

Asymmetrization of arbitrary metric (or norm) spaces

We can induce an asymmetrization of a metric space (X,D) via a P-asymmetrization of its free space $\mathcal{F} = \mathcal{F}(X,D)$

$$D_P(x,y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} \left(\phi(y) - \phi(x)\right) \quad \text{for all } x, y \in X.$$

Example: Consider $\mathbb{R}$ as a metric space with distance $D(x,y)=\|y-x\|$

Then. $L = \mathrm{Lip}_0(\mathbb{R})$

i) Asymmetrization through $L_+ = \{\phi \in L : \phi \geq 0\}$

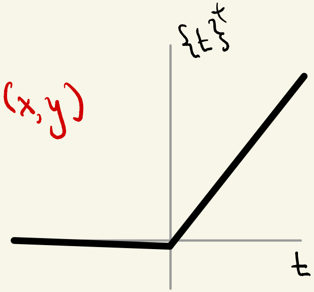
$$D_+(x,y) = \|\delta(y)-\delta(x)\|_{F_+} := \sup_{\substack{\|\phi\|_L \leq 1 \\ \phi \geq 0}} (\phi(y)-\phi(x)) \quad \left(\leq \max\{|y-x|, |y|\}\right)$$

Then if $0 < x < y$, we have $D_+(x,y) = |y-x|$ (Take $\phi_x(t) = |t|$).

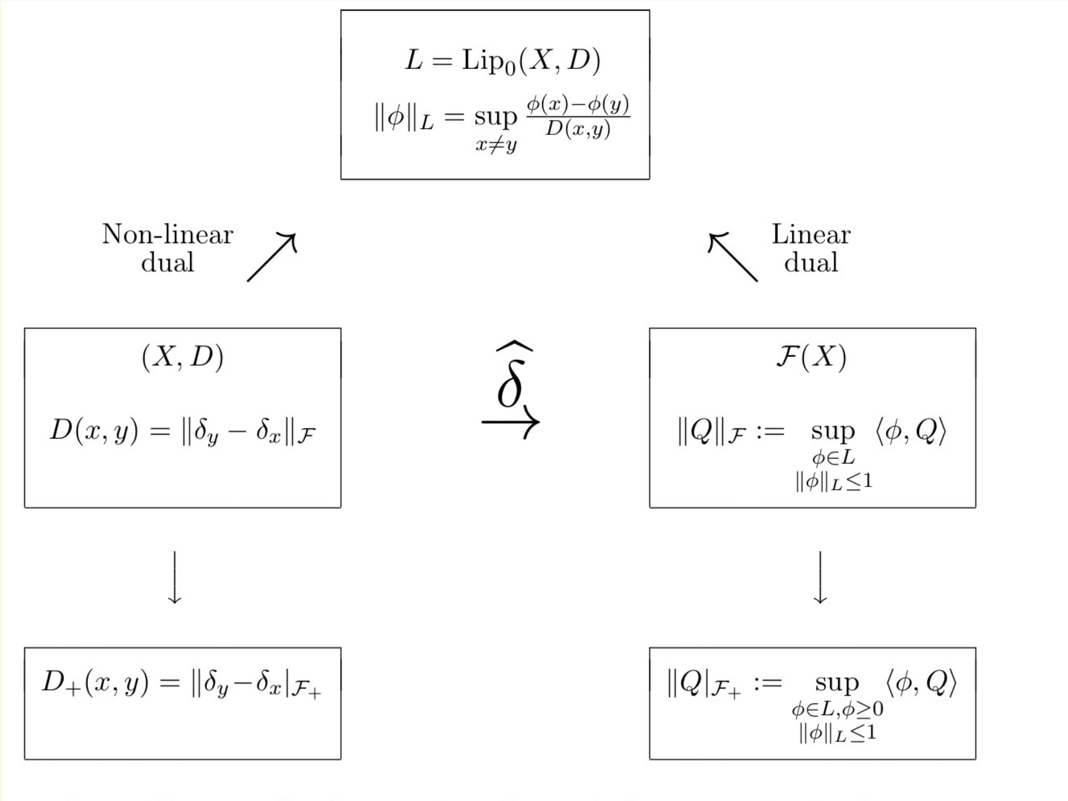
Thus $D_+(1,n) = n-1$ but $D_+(n,1) = 1$ for all $n \geq 2$

ii) Asymmetrization through $P = \{\phi \in L : \phi' \geq 0\}$

$$D_+(x,y) = \sup_{\substack{\|\phi\|_L \leq 1 \\ \text{increasing}}} [\phi(y)-\phi(x)] = \max\{y-x, 0\} = u(y-x) = d_u(x,y)$$



Asymmetrization vs symmetrization



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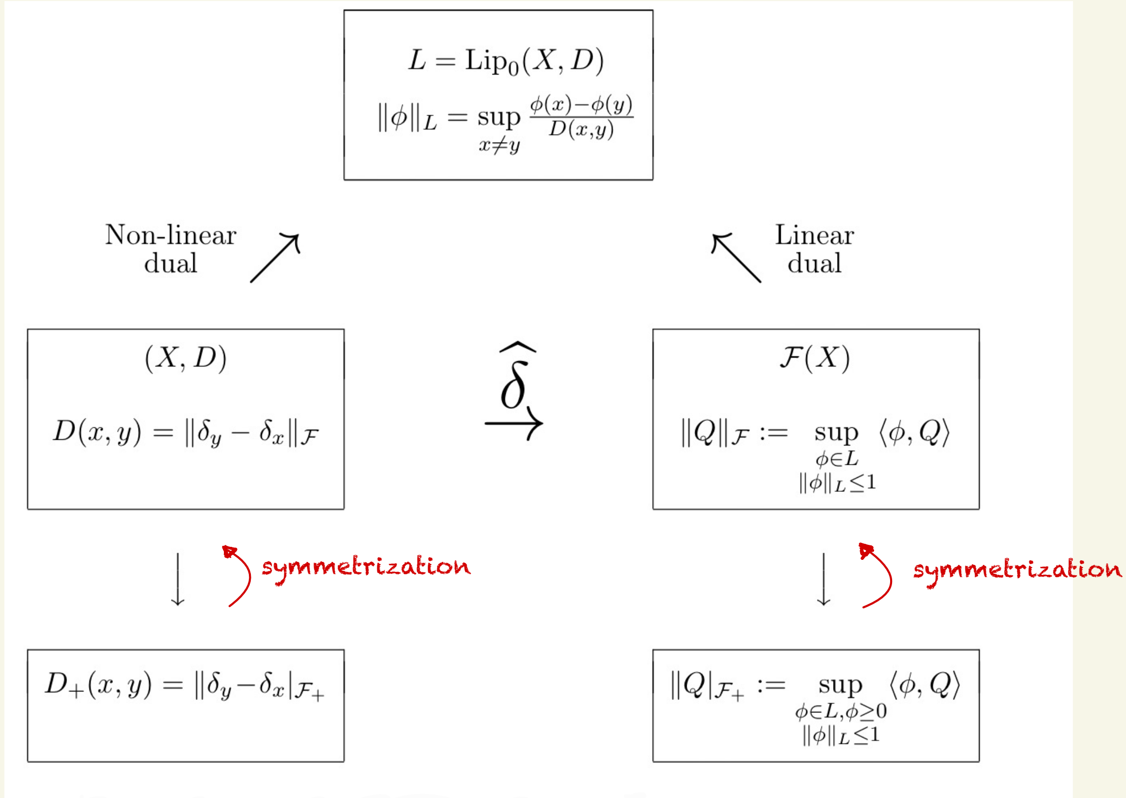
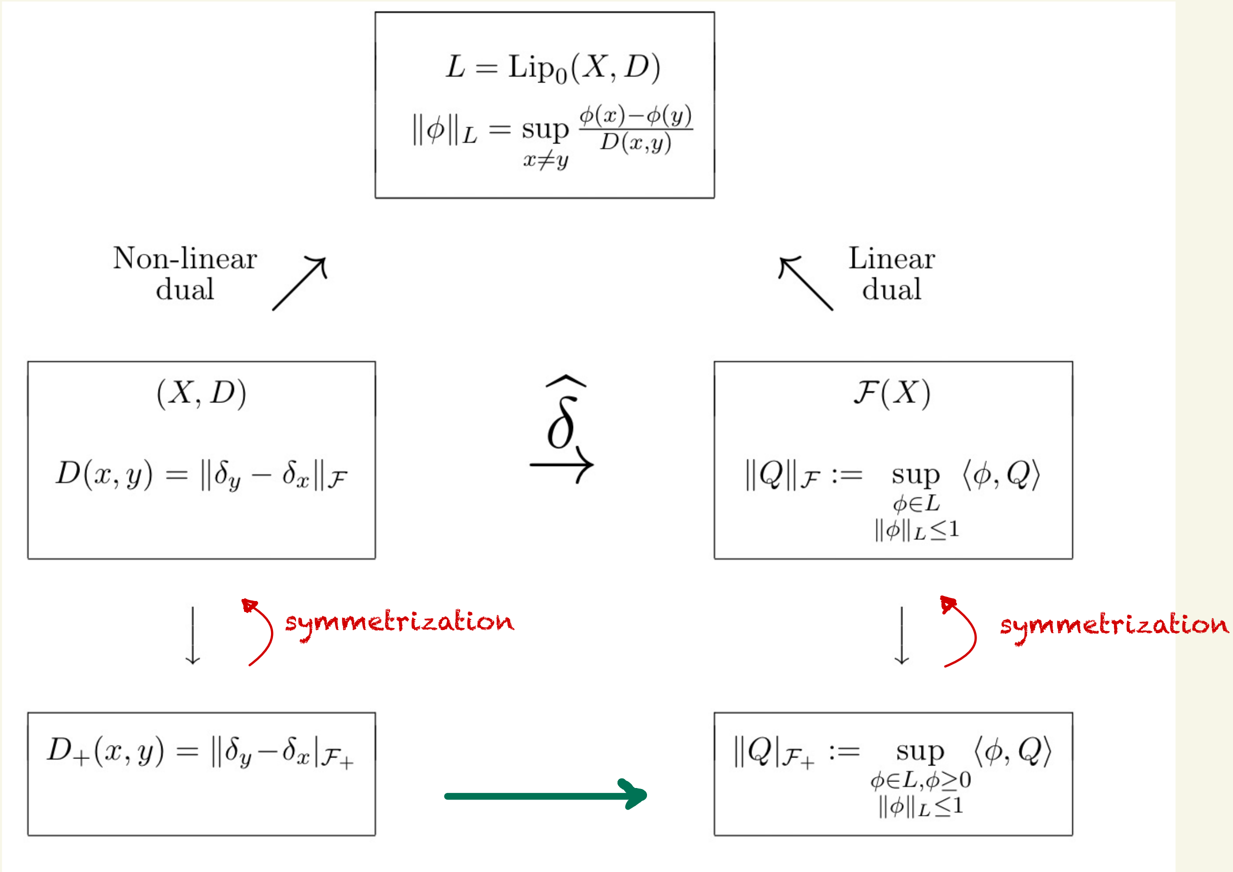


The spaces $\mathrm{Lip}_0(X,D)$ and $\mathrm{Lip}_0(X, D_+^s)$ are linearly isomorphic

Asymmetrization vs symmetrization



Semi-Lipschitz functions

(X, d) quasi-metric space

Definition (semi-Lipschitz function)

Let (X, d) be a quasi-metric space. A function $f : X \to \mathbb{R}$ is said to be **semi-Lipschitz** if there exists $L \geq 0$ such that for any $x, y \in X$

$$\mathbf{f(x) - f(y) \leq Ld(y, x).}$$

Semi-Lipschitz functions are:

- Lipschitz in (X, D) where $D = d^s$ (symmetrized distance)
- upper semicontinuous in (X, d) (forward topology)

Admit McShane extensions, AMSL extensions...

Example (Sorgenfrey line)

Consider $\mathbb{R}$ equipped with the quasi-distance d:

$$d(x,y) = \begin{cases} y-x, & \text{if } y \geq x \\ 1, & \text{if } y < x \end{cases}$$

The function $\boxed{f(x) = d(0, x)}$ is semi-Lipschitz:

$$f(x)-f(y) = d(0, x) - d(0, y) \lesssim d(y, x)$$

but it is not Lipschitz (ie. $-f$ is not semi-Lipschitz) :

Indeed, for every $r > 0$ the quantity

$$(-f)(0) - (-f)(-r) = f(-r) - f(0) = d(0, -r) = 1$$

cannot be controlled by $d(-r, 0) = (0-(-r)) = r$

Nonlinear dual of a quasi-metric space.

semi-Lipschitz constant

$$\|f|_{SL} := \sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y,x)}$$

Definition

$$\mathrm{SLIP}(X) := \{f : X \to \mathbb{R} \text{ such that } f \text{ is semi-Lipschitz}\}.$$

$$\mathrm{SLIP}_0(X) := \{f \in \mathrm{SLIP}(X) \text{ such that } f(x_0) = 0\}.$$

$(\mathrm{SLIP}(X), \|\cdot|_{SL})$ and $(\mathrm{SLIP}_0(X), \|\cdot|_{SL})$ are normed cones.

Linear (conic) duality

Proposition

Let $(C, \|\cdot|)$ be a normed cone, and $f : C \to \mathbb{R}$ a linear functional. TFAE

1. *f is upper semi-continuous for the forward topology of C,*
2. *$f : C \to (\mathbb{R}, d_u)$ is forward-forward continuous,*
3. *there exists $M \geq 0$ such that $f(x) \leq M\|x| \ \forall x \in C$,*
4. *$f \in \mathrm{SLIP}_0(C, d_e)$, where d_e is the extended quasi-metric induced by $\|\cdot|$.*

dual cone

$$C^* := \{f : C \to \mathbb{R} \text{ such that } f \text{ is linear and forward usc}\}.$$

dual norm

$$\|f|^* := \sup_{\|x|\leq 1} f(x)$$

Lipschitz free spaces (Kalton-Godefroy)

(X,D) pointed metric space (with space point x_0)

$$Lip_0(X) = \{\phi : X \longrightarrow \mathbb{R} \text{ Lipschitz } ; \phi(x_0) = 0\}$$

$$\|\phi\|_L = \sup_{x \neq y} \frac{\phi(x) - \phi(y)}{D(y,x)}$$

$$\delta : (X,D) \longrightarrow (Lip_0(X), \|\cdot\|_L)^*$$

isometry

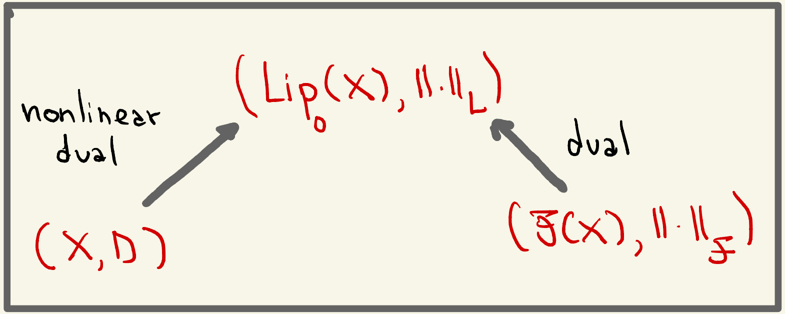
$$\delta(x) = \hat{x} \qquad \hat{x}(\phi) = \phi(x)$$

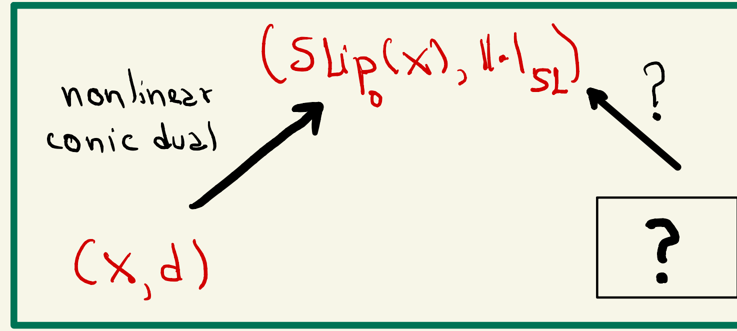
$$\mathcal{F}(X) = \overline{\mathrm{span}}\,(\delta(X)) \subset Lip_0(X)^*$$

$$\mathcal{F}(X)^* =_1 Lip_0(X)$$

$$\|Q\|_{\mathcal{F}} = \sup_{\|\phi\|_L \leq 1} \langle Q, \phi \rangle, \quad \forall Q \in \mathcal{F}(X)$$

Objective: define a Semi-Lipschitz free space





Main obstacle: the asymmetric analogue of $\mathrm{Lip}_0(X)$ is not a linear space, but a *cone*. Therefore, we had to:

1. Find a suitable framework to work with cones endowed with asymmetric norms.✓
2. Give a suitable notion of duality for these cones. ✓
3. Choose a compatible notion of completeness and completion. ✓

(X, d) quasi-metric space $\quad$ $SLip_0(X)$ cone of semi-Lipschitz functions

Proposition

For each $x \in X$, the evaluation functional $\hat{x} : \mathrm{SLIP}_0(X) \to \mathbb{R}$ defined by $\hat{x}(f) = f(x)$ belongs to the dual cone $(\mathrm{SLIP}_0(X), \|\cdot|_{SL})^*$.

In addition, $-\hat{x}$ also belongs to $(SLIP_0(X), \|.\|_{SL})^*$

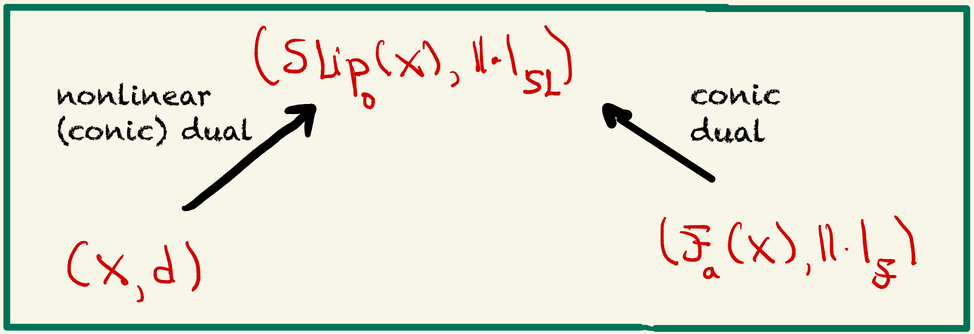
↓

The cone $(SLIP_0(X), \|.\|_{SL})^*$ contains $\mathrm{span}\{\hat{x} : x \in X\}$ as a linear subspace

Definition (Semi-Lipschitz free space)

The semi-Lipschitz free space over (X,d), denoted $\mathcal{F}_a(X)$, is the (unique) bi-completion of the asymmetric (normed) space $(\mathrm{span}\{\hat{x} : x \in X\}, \|\cdot|_{SL})^*$.

The dual cone of $\mathcal{F}_a(X)$ is isometrically isomorphic to $(SLIP_0(X), \|.\|_{SL})^*$.



Universal property

Let (X,d) *and* (Y,ρ) *be* ***pointed quasi-metric spaces****, and* $f \in \mathrm{SLIP}_0(X,Y)$*. Then there is a* ***unique linear map*** $T_f : \mathcal{F}_a(X) \to \mathcal{F}_a(Y)$ ***such that*** $T_f \circ \delta_X = \delta_Y \circ f$*, i.e. the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ \mathcal{F}_a(X) & \underset{T_f}{\dashrightarrow} & \mathcal{F}_a(Y) \end{array}$$

commutes, and $\|T_f| = \|f|_S$.

Relation to asymmetrizations

(X,D) metric space

$$L = (\mathrm{Lip}_0(X,D), \|\cdot\|_L)$$

$(\mathcal{F}(X,D), \|\cdot\|_{\mathcal{F}})$

(P-asymmetrization)

$d = D_P$

Recall: P generating closed convex cone of L (ie. $X = P - P$)

satisfying: $\forall \phi \in L,\ \exists \phi_1, \phi_2 \in P:$

$$\begin{cases} \phi = \phi_1 - \phi_2, \\ \max\{\|\phi_1\|_L, \|\phi_2\|_L\} \leq \|\phi\|_L \leq \|\phi_1\|_L + \|\phi_2\|_L. \end{cases}$$

↓

(X,d) quasi-metric space

$$SL = (\mathrm{SLip}_0(X, D_P), \|\cdot|_{SL})$$

$(\mathcal{F}_a(X,d), \|\cdot|_{\mathcal{F}_a})$

$$F = \operatorname{span}\{\delta(x) : x \in X\} \subset SL^*, \qquad \widehat{F} = \operatorname{span}\{\widehat{\delta}(x) : x \in X\} \subset L^*$$

$(X,d) \longmapsto \mathcal{F}_a(X,d)$ $\quad$ $(X,D) \longmapsto \mathcal{F}(X,D)$.

$$\begin{array}{ccccc}
F = \operatorname{span}(\delta(X)) & \sqsubseteq & \mathcal{F}_a(X) & \sqsubseteq & (SL)^* \\
\updownarrow & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}_a^s}\text{-dense} & & \\
\widehat{F} = \operatorname{span}(\widehat{\delta}(X)) & \sqsubseteq & \mathcal{F}(X) & \sqsubseteq & L^* \\
 & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}}\text{-dense} & &
\end{array}$$

Lemma (Isometric injection of P into SL). *For every metric space (X, D) and every P-asymmetrization (X, D_P):*

(i) *there exists an isometric injection of P into SL;*
(ii) *there is a non-expansive injection of SL into L.*

Proposition ($\|\cdot\|_{\mathcal{F}}$ is equivalent to the symmetrization of $\|\cdot|_{\mathcal{F}_a}$). *For any $Q \in F$,*

$$\boxed{\begin{aligned}\max\{\|Q|_{\mathcal{F}_a}, \|-Q|_{\mathcal{F}_a}\} \le \|\widehat{Q}\|_{\mathcal{F}} &\le \|\widehat{Q}|_{\mathcal{F}_P} + \|-\widehat{Q}|_{\mathcal{F}_P} \\ &\le 2\max\{\|Q|_{\mathcal{F}_a}, \|-Q|_{\mathcal{F}_a}\}.\end{aligned}}$$

Theorem (Compatibility). *Let (X, D) be a metric space with a P-asymmetrization. Then the symmetrizations of $(\mathcal{F}(X, D), \|\cdot|_{\mathcal{F}_P})$ and of $(\mathcal{F}_a(X), \|\cdot|_{\mathcal{F}_a})$ are both isomorphic to $(\mathcal{F}(X, D), \|\cdot\|_{\mathcal{F}})$.*

Examples

* A 3-point quasi-metric space

$X = \{x_0, x_1, x_2\}$ ρ quasi-distance $\mathcal{F}_a(X,\rho) = (\mathbb{R}^2, \|\cdot\|_{\mathcal{F}_a})$

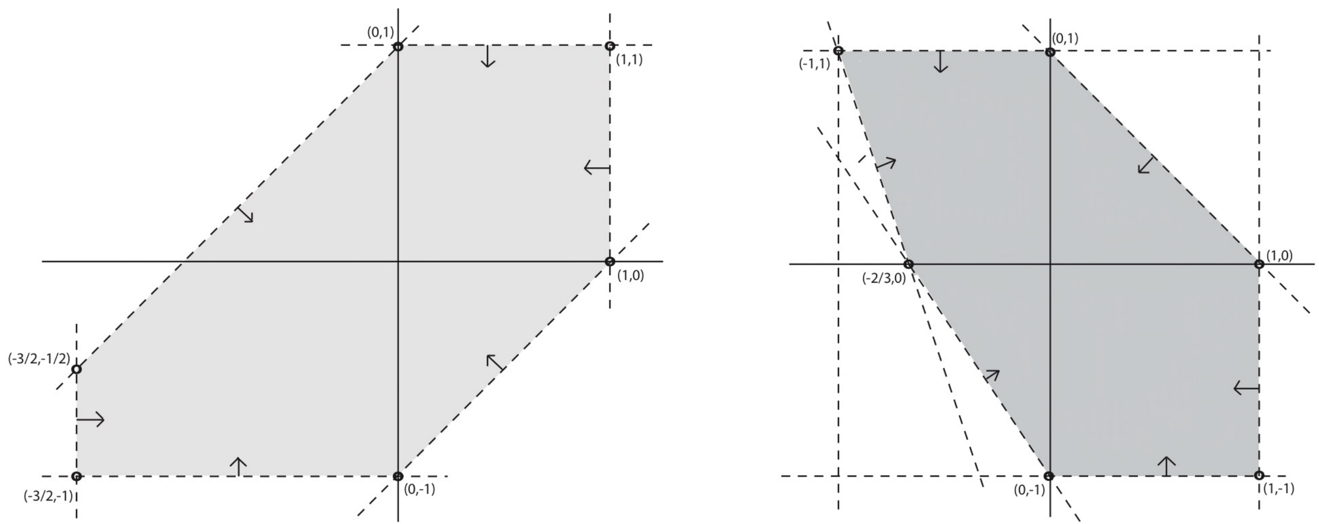


Fig. Representation of the unit ball of $\mathrm{SLip}_0(X,\rho)$ and its asymmetric polar, respectively, with $X = \{x_0, x_1, x_2\}$, $\rho(x_1, x_0) = \frac{3}{2}$ and $\rho(x_i, x_j) = 1$ for $i \neq j$ with $(i,j) \neq (1,0)$

* $\mathbb{N}$ as a quasi-metric space

$$d(n,m) = \begin{cases} 1 & \text{if } m \notin \{0,n\}, \\ 0 & \text{if } m \in \{0,n\}. \end{cases}$$

$$\mathcal{F}_a(\mathbb{N},d) =_1 (\ell^1(\mathbb{N}), \|\cdot\|_{1,+})$$

$$\|x\|_{1,+} = \sum_{n=0}^{\infty} \max\{x_n, 0\}.$$

* The quasi-metric space $(\mathbb{R}, d_u)$

$$d_u(x,y) = \max\{y-x, 0\}$$

$$\mathcal{F}_a(\mathbb{R}, d_u) =_1 (\mathcal{L}^1(\mathbb{R}), \|\cdot\|_{1,+})$$

$$\|f\|_{1,+} = \int_{\mathbb{R}} \{f(t)\}^+ \, dt$$

* Canonical asymmetrization of subsets of $\mathbb{R}$-trees

DEFINITION (ℝ-tree). An ℝ-*tree* is a metric space T satisfying:

(i) for any $x, y \in T$, there exists a unique isometry $\phi =: \phi_{xy}$ of the closed interval $[0, d(x, y)]$ into T such that $\phi(0) = x$ and $\phi(d(x, y)) = y$ (the range of this isometry is called the *segment* and is denoted by $[x, y]$);

(ii) any one-to-one continuous mapping $\varphi : [0, 1] \to T$ has the same range as the isometry $\phi_{a,b}$ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

PROPOSITION *Let (X, D) be a subset of an ℝ-tree. Then there exists a canonical asymmetrization D_P of D such that the symmetrization of the semi-Lipschitz free space $\mathcal{F}_a(X, D_P)$ is isometrically isomorphic to $\mathcal{F}(X, D)$.*

The corresponding semi-Lipschitz free spaces are isometrically isomorphic to $(\ell^1(\Gamma), \|\cdot|_{1,+})$ for some set Γ

STUDIA MATHEMATICA 261 (1) (2021)

Asymmetric free spaces and canonical asymmetrizations

by

Aris Daniilidis (Santiago), Juan Matías Sepulcre (Alicante)
and Francisco Venegas M. (Santiago)

Abstract. A construction analogous to that of Godefroy–Kalton for metric spaces allows one to embed isometrically, in a canonical way, every quasi-metric space (X,d) in an asymmetric normed space $\mathcal{F}_a(X,d)$ (its quasi-metric free space, also called asymmetric free space or semi-Lipschitz free space). The quasi-metric free space satisfies a universal property (linearization of semi-Lipschitz functions). The (conic) dual of $\mathcal{F}_a(X,d)$ coincides with the non-linear asymmetric dual of (X,d), that is, the space $\mathrm{SLip}_0(X,d)$ of semi-Lipschitz functions on (X,d), vanishing at a base point. In particular, for the case of a metric space (X,D), the above construction yields its usual free space. On the other hand, every metric space (X,D) naturally inherits a canonical asymmetrization coming from its free space $\mathcal{F}(X)$. This gives rise to a quasi-metric space (X,D_+) and an asymmetric free space $\mathcal{F}_a(X,D_+)$. The symmetrization of the latter is isomorphic to the original free space $\mathcal{F}(X)$. The results of this work are illustrated with explicit examples.

Contents

2020 *Mathematics Subject Classification*: Primary 46B20; Secondary 39B82, 46A22.
Key words and phrases: free space, canonical asymmetrization, semi-Lipschitz functions, quasi-metric space.

More details:

A. Daniilidis, J.M.Sepulcre, F. Venegas M.
Asymmetric free spaces
and canonical asymmetrizations
Studia Math. 261 (2021), 55–102

available also in ArXiv and in my webpage
(google: Aris Daniilidis, TU Wien)

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